Question:

1. Existe-t-il une fonction $Q(x, y)$ telle que la fonction $z = x + iy \longrightarrow x^2 + y^2 + iQ(x, y)$ soit holomorphe?
2. Même question pour $\ln(x^2 + y^2)$
3. Plus généralement, étant donné $P(x, y)$, peut-on trouver $Q(x, y)$ tel que la fonction

$$x + iy \mapsto P(x, y) + iQ(x, y)$$

soit une fonction holomorphe?

Théorème

Si P, Q sont de classe C^2 et $P + iQ$ est holomorphe, alors

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0.$$

De même,

$$\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} = 0.$$

(On dit que P et Q sont des fonctions harmoniques)

$$\begin{aligned}
\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial Q}{\partial x}\right) \\
&= \frac{\partial^2 Q}{\partial x \partial y} - \frac{\partial^2 Q}{\partial y \partial x} \\
&= 0
\end{aligned}$$

Exemples

- $P(x,y) = x^2 - y^2$ est harmonique: $\dfrac{\partial^2 P}{\partial x^2} + \dfrac{\partial^2 P}{\partial y^2} = 2 - 2 = 0$
- $Q(x,y) = 2xy$ est harmonique: $\dfrac{\partial^2 Q}{\partial x^2} + \dfrac{\partial^2 Q}{\partial y^2} = 0 + 0 = 0$
- $R(x,y) = x^3 - 3xy^2$ est harmonique:

$$\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} = \frac{\partial}{\partial x}\left(3x^2 - 3y^2\right) + \frac{\partial}{\partial y}(-6xy) = 6x - 6x = 0.$$

- $H(x,y) = x^2y^2$ n'est pas harmonique:

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{\partial}{\partial x}\left(2xy^2\right) + \frac{\partial}{\partial y}\left(2x^2y\right) = 2y^2 + 2x^2 \neq 0.$$

Exercice

Trouver les conjugés harmoniques Q des fonctions P suivantes.
Trouver la fonction complexe $f = P + iQ$ correspondante.

Exercice

$P(x, y) = x^3 - 3xy^2.$

P est harmonique:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \frac{\partial}{\partial x}\left(3x^2 - 3y^2\right) + \frac{\partial}{\partial y}(-6xy) = 6x - 6x = 0.$$

Exercice

$\underline{P(x,y) = x^3 - 3xy^2.}$

$\dfrac{\partial Q}{\partial y} = \dfrac{\partial P}{\partial x} = 3x^2 - 3y^2.$ Donc

$$Q(x,y) = 3x^2y - y^3 + g(x).$$

$\dfrac{\partial Q}{\partial x} = -\dfrac{\partial P}{\partial y} = 6xy$ donc $6xy + g'(x) = 6xy$, $g' = 0$, $g = C$ et

$$Q(x,y) = 3x^2y - y^3 + C.$$

On a alors

$$P(x,y) + iQ(x,y) = (x^3 - 3xy^2) + i(3x^2y - y^3 + C) = (x+iy)^3 + iC$$

$$f(z) = f(x+iy) = z^3 + C'$$

Exercice

$P(x, y) = e^x \cos y.$

P est harmonique:

$$\begin{aligned} \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} &= \frac{\partial}{\partial x}(e^x \cos y) + \frac{\partial}{\partial y}(-e^x \sin y) \\ &= e^x \cos y - e^x \cos y = 0. \end{aligned}$$

Exercice

$P(x,y) = e^x \cos y.$

$\frac{\partial Q}{\partial y} = \frac{\partial P}{\partial x} = e^x \cos y.$ Donc

$$Q(x,y) = e^x \sin y + g(x).$$

$\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y}$ donc $e^x \sin y + g'(x) = e^x \sin y$, $g' = 0$, $g = C$ et

$$Q(x,y) = e^x \sin y + C.$$

On a alors

$$P(x,y)+iQ(x,y) = e^x \cos y+i(e^x \sin y+C) = e^x(\cos y+i \sin y)+iC.$$

$$f(z) = f(x+iy) = e^z + iC$$